A Differential Game Analysis of Free Entry Oligopoly with Sticky Prices

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Abstract

This study analyzes a free entry oligopoly under general demand and cost functions via a differential game approach. We consider an open-loop dynamic oligopoly and a memoryless closed-loop dynamic oligopoly with sticky prices. The results show under some assumptions that the number of firms at the steady state in the open-loop dynamic oligopoly is fewer than the number of firms in the static equilibrium, and that the number of firms in the memoryless closed-loop dynamic oligopoly is fewer than the number of firms in the open-loop dynamic oligopoly.

Keywords: free entry oligopoly; differential game; general demand function; general cost function; open-loop; memoryless closed-loop

JEL Classifications: C73, D43, L13.

1. Introduction

In this paper we analyze a free entry dynamic oligopoly under general demand and cost functions by a differential game approach. Mainly we show that the number of firms at the steady state in the open-loop dynamic oligopoly is smaller than that in the static equilibrium, and that in the memoryless closed-loop dynamic oligopoly is smaller than that in the open-loop dynamic oligopoly.

There are many studies of dynamic oligopoly by differential game approach, for example, Simaan and Takayama (1978), Fershtman and Kamien (1987), Cellini and Lambertini (2004) and Cellini and Lambertini (2007) about behaviors of firms and market structures with dynamics of sticky prices in an oligopoly with a homogeneous good or differentiated goods, Cellini and Lambertini (2003a) and Cellini and Lambertini (2003b) about advertising investment with dynamics of accumulated advertising effects in an oligopoly with a homogeneous good or differentiated goods, Cellini and Lambertini (2005) and Cellini and Lambertini (2011) about R&D investment with dynamics of accumulated cost reducing effects in an oligopoly with a homogeneous good or differentiated goods, Fujiwara (2006) about a Stackelberg duopoly, Fujiwara (2008) about competitiveness of markets in an oligopoly with renewable resource. For a comprehensive survey, see Dockner et al. (2000) and Lambertini

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(2018). In particular, Cellini and Lambertini (2004) studied an oligopoly in which firms produce a homogeneous good. However, most of these studies including Cellini and Lambertini (2004) used a model of specific (linear or quadratic) demand and cost functions. These are limited assumptions. We study a steady state of free entry oligopoly with a homogeneous good under general demand and cost functions. In the next section we present a model and assumptions of this paper. In Section 3 we consider an open-loop solution of a differential game analysis of free entry oligopoly. In Section 4 we examine a memoryless closed-loop solution.

In Propositions 1 and 2 we show the following results.

1. If the inverse demand function is concave or its second order derivative is not so large when it is strictly convex, then the number of firms at the steady state of the free entry open-loop dynamic oligopoly is decreasing with respect to discount rate, and it is increasing with respect to the speed of price dynamics. Thus, under this assumption the number of firms at the steady state in the free entry open-loop dynamic oligopoly is smaller than that at the static equilibrium.

2. The number of firms at the steady state of the memoryless closed-loop free entry oligopoly is smaller than the number of firms at the steady state of the open-loop free entry oligopoly.

2. The Model

Our model of dynamic oligopoly is based on the model in Simaan and Takayama (1978), Fershtman and Kamien (1987), and Cellini and Lambertini (2004). We generalize linear demand and cost functions in their models to general demand and cost functions. There is a symmetric oligopoly where, at any $t \in [0, \infty)$, n firms, and firms $1, 2, \dots, n$ produce a homogeneous good. Firms maximize their discounted profits. Let $x_i(t)$, $i \in \{1, 2, \dots, n\}$, be the outputs of the firms, $p(t)$ be the price of the good at t.

The inverse demand function is

$$\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)).$$

We assume

$$\hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t)) < 0.$$

The cost function of Firm i, $i \in \{1,2,\dots,n\}$, is

$$c(x_i(t)), \qquad i \in \{1,2,\dots,n\}.$$

All firms have the same cost functions. It satisfies $c'(x_i(t)) > 0$ and $c''(x_i(t)) > 0$, that is, it is strictly increasing and strictly convex. The instantaneous profit of Firm i, is

$$\pi_i(t) = x_i(t)p(t) - c(x_i(t)), \qquad i \in \{1,2,\dots,n\}.$$

The price of the good evolves according to the following dynamics.

$$\frac{dp(t)}{dt} = s[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)], \qquad s > 0, \qquad p(0) > 0. \qquad (1)$$

s is the speed of price adjustment, and it is an inverse measure of price stickiness. There are black-boxing menu costs or other similar mechanism behind this model (see, for example, Lambertini, 2018). The problem of Firm i is

$$\max_{x_i(t)} \int_0^\infty e^{-\rho t}[x_i(t)p(t) - c(x_i(t))]dt,$$

subject to (1). $\rho > 0$ is the discount rate.

The present value Hamiltonian function of Firm i, $i \in \{1,2,\dots,n\}$, is

$$\mathcal{H}_i(t) = e^{-\rho t}\{x_i(t)p(t) - c(x_i(t)) + \lambda_i(t)s[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)]\}.$$

The current value Hamiltonian function of Firm i, $i \in \{1,2,\dots,n\}$, is

$$\widehat{\mathcal{H}}_i(t) = e^{\rho t}\mathcal{H}_1(t)$$

$$= x_i(t)p(t) - c(x_i(t)) + \lambda_i(t)s[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)]\}.$$

Let

$$\mu_i(t) = e^{-\rho t}\lambda_i(t), \qquad i \in \{1,2,\dots,n\}.$$

$\mu_i(t)$ is the costate variable.

The free entry condition is written as follows.

$$\int_0^\infty e^{-\rho t}[x_i(t)p(t) - c(x_i(t))]dt = 0.$$

At the steady state the number of firms as well as the price and the output of each firm are constant. Denoting them by n, p, x, the free entry condition is

$$px - c(x) = 0. \qquad (2)$$

This holds in all cases. When all $x_i(t)'s$ are equal,

$$\frac{\partial p}{\partial n(t)} = p'\ x_i(t), \qquad \frac{\partial p'}{\partial n(t)} = p''x_i(t).$$

3. Open-loop Solution

We consider an open-loop solution according to the analyses in Cellini and Lambertini (2003), Cellini and Lambertini (2004), Cellini and Lambertini (2007) and Lambertini (2018). The first order condition for Firm i is

$$\frac{\partial \widehat{\mathcal{H}}_i(t)}{\partial x_i(t)} = p(t) + \hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - c'(x_i(t)) = 0. \qquad (3)$$

The second order condition is

$$\frac{\partial^2 \widehat{\mathcal{H}}_i(t)}{\partial x_i(t)^2} = \hat{p}''(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - c''(x_i(t)) < 0. \quad (4)$$

The adjoint condition is

$$-\frac{\partial \widehat{\mathcal{H}}_i(t)}{\partial p(t)} = -x_i(t) + \lambda_i(t)s = \frac{\partial \lambda_i(t)}{\partial t} - \rho\lambda_i(t), \qquad i \in \{1,2,\dots,n\}.$$

This means

$$\frac{\partial \lambda_i(t)}{\partial t} = (\rho + s)\lambda_i(t) - x_i(t), \qquad i \in \{1,2,\dots,n\}.$$

Differentiating (3) with respect to time, we get

$$[c''(x_i(t)) - \hat{p}''\lambda_i(t)s]\frac{dx_i(t)}{dt} = \frac{dp(t)}{dt} + \hat{p}'\frac{\partial \lambda_i(t)}{\partial t}s$$
$$= \frac{dp(t)}{dt} + \hat{p}'s[(\rho + s)\lambda_i(t) - x_i(t)], \qquad i \in \{1,2,\dots,n\}. \quad (5)$$

We denote $\hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))$ by $\hat{p}'$, and $\hat{p}''(x_1(t) + x_2(t) + \cdots + x_n(t))$ by $\hat{p}''$.

At the steady state $\frac{dp(t)}{dt} = 0$, $\frac{dx_i(t)}{dt} = 0$ and $\frac{\partial \lambda_i(t)}{\partial t} = 0$ for $i \in \{1,2,\dots,n\}$. By symmetry of the oligopoly, at the steady state all $x_i(t)$'s and $\lambda_i(t)$ are equal. Denote $x_i(t)$, $p(t)$ and $\lambda_i(t)$ at the steady state by x^*, p^* and λ^*. Then, from (5)

$$(\rho + s)\hat{p}'s\lambda^* = \hat{p}'sx^*.$$

Substituting this into (3) yields

$$(\rho + s)\frac{\partial \widehat{\mathcal{H}}_i(t)}{\partial x_i(t)} = (\rho + s)[p^* - c'(x^*)] + \hat{p}'sx^* = 0. \quad (6)$$

This means

$$p^* - c'(x^*) > 0 \quad \text{and} \quad p^* - c'(x^*) + \hat{p}'x^* < 0. \quad (7)$$

By the free entry condition (2),

$$p^*x^* - c(x^*) = 0. \quad (8)$$

Denote the steady state value of $n(t)$ by n^*. Since $p^* = \hat{p}(n^*x^*)$, from (8)

$$\frac{dn^*}{dx^*} = -\frac{p^* + n^*\hat{p}'(n^*x^*)x^* - c'(x^*)}{\hat{p}' \cdot (x^*)^2} < 0 \quad (\text{from} \quad (7) \quad \text{and} \quad \hat{p}' < 0), \quad (9)$$

and

$$\frac{d(n^*x^*)}{dx^*} = n^* + x^*\frac{dn^*}{dx^*} = -\frac{p^* - c'(x^*)}{\hat{p}' \cdot (x^*)^2} > 0.$$

Equation (9) is the relation between n^* and x^* which satisfy the free entry condition. This is common to the open-loop and the closed-loop cases. The equilibrium of the static oligopoly is obtained by setting $\rho = 0$ (or $s \to +\infty$). Differentiating (6) with respect to ρ, we obtain

$$\frac{dx^*}{d\rho} = -\frac{p^* - c'(x^*)}{(\rho+s)\left[\hat{p}'\frac{d(n^*x^*)}{dx^*} - c''(x^*)\right] + \hat{p}'s + \hat{p}''sx^*\frac{d(n^*x^*)}{dx^*}}.$$

Similarly

$$\frac{dx^*}{ds} = -\frac{p^* - c'(x^*) + \hat{p}'x^*}{(\rho+s)\left[\hat{p}'\frac{d(n^*x^*)}{dx^*} - c''(x^*)\right] + \hat{p}'s + \hat{p}''sx^*\frac{d(n^*x^*)}{dx^*}}.$$

Since $\hat{p}' < 0$, $\frac{d(n^*x^*)}{dx^*} > 0$, $c''(x^*) > 0$, $p^* - c'(x^*) > 0$, $p^* - c'(x^*) + \hat{p}'x^* < 0$, if $\hat{p}'' \leq 0$ or $|\hat{p}''|$ is not so large even when $\hat{p}'' > 0$, we have $\frac{dx^*}{d\rho} > 0$ and $\frac{dx^*}{ds} < 0$. From (9)

$$\frac{dn^*}{d\rho} < 0, \qquad \frac{dn^*}{ds} > 0.$$

We have shown the following result.

Proposition 1 *If the inverse demand function is concave* ($\hat{p}'' \leq 0$) *or* $|\hat{p}''|$ *is not so large when it is strictly convex* ($\hat{p}'' > 0$)*, then the number of firms at the steady state in the free entry open-loop dynamic oligopoly is decreasing with respect to* ρ *(discount rate), and it is increasing with respect to* s *(speed of price adjustment). Thus, under this assumption the number of firms at the steady state in the free entry open-loop dynamic oligopoly is smaller than the number of firms at the static free entry equilibrium.*

Linear Demand and Quadratic Cost Example

Suppose that the inverse demand function is

$$\hat{p}(t) = a - \sum_{i=1}^{n} x_i(t). \tag{10}$$

a is a positive constant. Also suppose that the cost function of Firm i, $i \in \{1,2,\dots,n\}$, is

$$c(x_i(t)) = cx_i(t) + \frac{1}{2}x_i(t)^2 + f. \tag{11}$$

$f > 0$ is the fixed cost. Then, (6) and (8) are

$$(\rho+s)[a - n^*x^* - c - x^*] - sx^* = 0,$$

and

$$(a - n^*x^*)x^* - cx^* - \frac{1}{2}(x^*)^2 - f = 0.$$

From them we get

$$x^* = \sqrt{\frac{2f(\rho+s)}{\rho+3s}},$$

and

$$n^* = \frac{\sqrt{2}(a-c)\sqrt{f(\rho+3s)(\rho+s)} + 2f\rho + 4fs}{2f(s+\rho)}.$$

Differentiating x^* with respect to ρ and s yields

$$\frac{dx^*}{d\rho} = \frac{\sqrt{2f}s}{\sqrt{\rho+s}\sqrt{(\rho+3s)^3}} > 0,$$

and

$$\frac{dx^*}{ds} = -\frac{\sqrt{2f}\rho}{\sqrt{\rho+s}\sqrt{(\rho+3s)^3}} < 0.$$

4. Memoryless Closed-loop Solution

In this section, according to the analyses in Cellini and Lambertini (2003a),Cellini and Lambertini (2004), Cellini and Lambertini (2007) and Lambertini (2018) (p.65), we consider a memoryless closed-loop approach to a dynamic oligopoly. The current value Hamiltonian function, the first order condition and the second order condition for Firm i, $i \in \{1,2,\dots,n\}$, are the same as those in the open-loop case as follows;

$$\widehat{\mathcal{H}}_i(t) = x_i(t)p(t) - c(x_i(t)) + \lambda_i(t)s[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)]\}.$$

$$\frac{\partial \widehat{\mathcal{H}}_i(t)}{\partial x_i(t)} = p(t) + \hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - c'(x_i(t)) = 0, \tag{12}$$

and

$$\frac{\partial^2 \widehat{\mathcal{H}}_i(t)}{\partial x_i(t)^2} = \hat{p}''(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - c''(x_j(t)) < 0. \tag{13}$$

The adjoint condition for Firm $i \in \{1,2,\dots,n\}$ is different from that in the open-loop case. In the memoryless closed-loop case it is written as

$$-\frac{\partial \widehat{\mathcal{H}}_i(t)}{\partial p(t)} - \sum_{j\neq i} \frac{\partial \widehat{\mathcal{H}}_i(t)}{\partial x_j(t)} \frac{\partial x_j(t)}{\partial p(t)} = \frac{\partial \lambda_i(t)}{\partial t} - \rho\lambda_i(t). \tag{14}$$

The term in (14)

$$-\sum_{j\neq i}\frac{\partial\widehat{\mathcal{H}}_i(t)}{\partial x_j(t)}\frac{\partial x_j(t)}{\partial p(t)}$$

takes into account the interaction between the control variable of the firms other than Firm i and the current level of the state variable.

We have

$$\frac{\partial\widehat{\mathcal{H}}_i(t)}{\partial x_j(t)}=\hat{p}'\lambda_i(t)s.$$

About $\frac{\partial x_j(t)}{\partial p(t)}$ from the first order condition,

$$\frac{\partial x_j(t)}{\partial p(t)}=\frac{1}{c''(x_j(t))-\hat{p}''\lambda_j(t)s}.$$

Therefore, (14) is rewritten as

$$-\frac{\partial\widehat{\mathcal{H}}}{\partial p(t)}-\sum_{j\neq i}\frac{\partial\widehat{\mathcal{H}}_i(t)}{\partial x_j(t)}\frac{\partial x_j(t)}{\partial p(t)}$$
$$=-x_i(t)+\lambda_i(t)s-\hat{p}'\lambda_i(t)s\sum_{j\neq i}\frac{1}{c''(x_j(t))-\hat{p}''\lambda_j(t)s}=\frac{\partial\lambda_i(t)}{\partial t}-\rho\lambda_i(t).$$

This means

$$\frac{\partial\lambda_i(t)}{\partial t}=(\rho+s)\lambda_i(t)-x_i(t)-\hat{p}'\lambda_i(t)s\sum_{j\neq i}\frac{1}{c''(x_j(t))-\hat{p}''\lambda_j(t)s}.$$

Differentiating (12) with respect to time, we get

$$[c''(x_i(t))-\hat{p}''\lambda_i(t)s]\frac{dx_i(t)}{dt}=\frac{dp(t)}{dt}+\hat{p}'\frac{\partial\lambda_i(t)}{\partial t}s$$
$$=\frac{dp(t)}{dt}+\hat{p}'s[(\rho+s)\lambda_i(t)-x_i(t)]-(\hat{p}'s)^2\lambda_i(t)\sum_{j\neq i}\frac{1}{c''(x_j(t))-\hat{p}''\lambda_j(t)s}.$$

At the steady state $\frac{dp(t)}{dt}=0$, $\frac{dx_i(t)}{dt}=0$ and $\frac{\partial\lambda_i}{\partial t}=0$ for $i\in\{1,2,\dots,n\}$. By symmetry of the oligopoly, at the steady state all $x_i(t)$'s are equal and all $\lambda_i(t)$'s are equal. Denote $x_i(t)$, $p(t)$ and $\lambda_i(t)$ at the steady state by x^{**}, p^{**} and λ^{**}. Then, we obtain

$$\hat{p}'s\left[(\rho+s)-\hat{p}'s(n-1)\frac{1}{c''(x^{**})-\hat{p}''\lambda^{**}s}\right]\lambda^{**}=\hat{p}'sx^{**}.$$

Substituting this into (12),

$$\left[(\rho+s)-\hat{p}'s(n-1)\frac{1}{c''(x^{**})-\hat{p}''\lambda^{**}s}\right]\frac{\partial\widehat{\mathcal{H}}_i(t)}{\partial x_i(t)}$$
$$=\left[(\rho+s)-\hat{p}'s(n-1)\frac{1}{c''(x^{**})-\hat{p}''\lambda^{**}s}\right][p^{**}-c'(x^{**})]+\hat{p}'sx^{**}=0. \tag{15}$$

The free entry condition is as follows;

$$p^{**}x^{**}-c(x^{**})=0. \tag{16}$$

Denote the steady state value of $n(t)$ by n^{**}. Suppose that $x_i(t)=x^{**}$. x^{**} and n^{**} satisfy (15). By the second order condition (13),

$$c''(x^{**})-\hat{p}''\lambda^{**}s>0, \tag{17}$$

From $\hat{p}'<0$, (15) and (17),

$$p^{**}-c'(x^{**})>0.$$

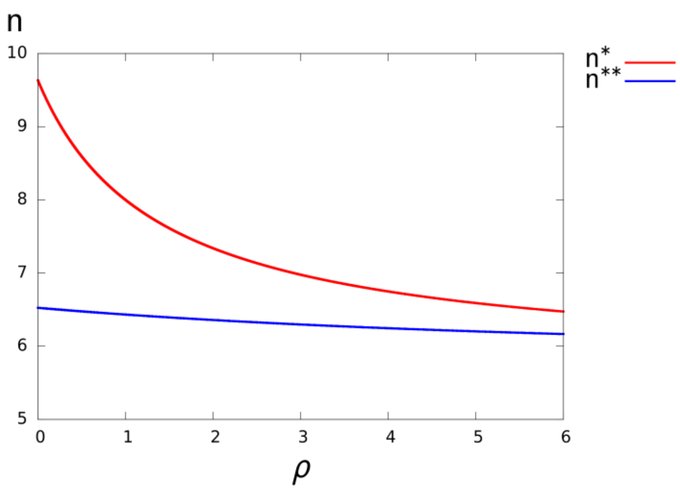
Thus, we have

$$(\rho+s)[p^{**}-c'(x^{**})]+\hat{p}'sx^{**}=\hat{p}'s(n-1)\frac{1}{c''(x^{**})-\hat{p}''\lambda^{**}s}[p^{**}-c'(x^{**})]<0.$$

Therefore, the left-hand side of (3) or (6), which are the first order conditions in the open-loop case, is negative. By the second order condition in the open-loop case (4) we get

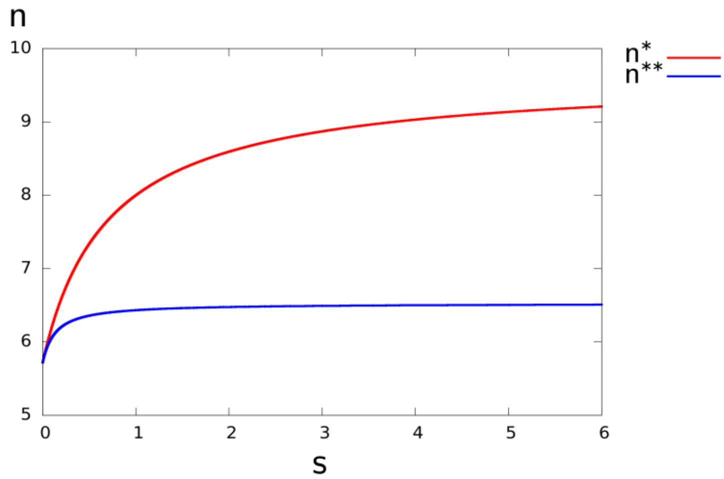
$$x^{**}>x^{*}.$$

Then, from (9) we have $n^{*}>n^{**}$. The following result has been shown.



Note: The upper line depicts n*, and the lower line depicts n**.

Figure 1: The Numbers of Firms in Open-loop, Closed-loop and ρ



Note: The upper line depicts n*, and the lower line depicts n**.

Figure 2: The Numbers of Firms in Open-loop, Closed-loop and s

Proposition 2 *The number of firms at the steady state of the memoryless closed-loop free entry oligopoly is smaller than the number of firms at the steady state of the open-loop free entry oligopoly.*

Linear Demand and Quadratic Cost Example

Suppose that the inverse demand function is (10), and the cost function of Firm i, $i \in \{1,2,\dots,n\}$, is (11). Then, since $c''(x_j(t)) = 1$ and $\hat{p}'' = 0$, (15) is reduced to

$$[(\rho + s) + (n-1)s][a - (n+1)x^{**} - c] - sx^{**} = 0.$$

(16) is written as follows;

$$(a - n^{**}x^{**})x^{**} - cx^{**} - \frac{1}{2}(x^{**})^2 - f = 0.$$

Solutions of them are complicated, and so we present graphical representations in Figure 1 and Figure 2. Assuming $f = 4$, $a = 20$, $c = 1$, $s = 1$ or $f = 4$, $a = 20$, $c = 1$, $\rho = 1$, they depict the relations of n^*, n^{**}, ρ and s.

5. Concluding Remark

We have studied the problem of free entry Cournot oligopoly by a differential game approach under general demand and cost functions with a homogeneous good. In the future research we will study free entry oligopoly with differentiated goods, or monopolistic competition.

How many firms enter the market in an oligopoly is important for the design of competition policy by the government. If firms make their decisions intertemporally so as to maximize the discounted profits, the analysis of a dynamic free entry oligopoly is an important theme of research.

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