A New Explanation of Arms Races in the Third World: A Differential Game Model

Cheng-Te Lee*

This paper examines the stability of arms races in the Third World countries and assumes that the utility function is separable between the consumption and the weapon stocks. We find that the military expenditures and the weapon stocks will exhibit stability and overshooting takes place for the optimal control models. Moreover, we prove that the differential game model has an unstable equilibrium, a result opposite of Deger and Sen (1984).

Keywords: arms race, differential game model
JEL classification: H5

1 Introduction

The stability of armament races has in recent years received considerable attention. Classical arms race models which have a stable equilibrium generally deal with East-West conflicts or superpower arms races (see e.g., Richardson, 1960; Brito, 1972; Intriligator, 1975; Intriligator and Brito, 1976; Simaan and Cruz, 1975; Melese and Michel, 1991). These models are not applicable to the Third World countries, because of two characteristics of the Third World countries. First, given stringent resource constraints, the Third World countries have to be extra careful about the allocation of resources between defense and civilian sectors. Second, in the Third World countries, most conflicts on a regional scale have been between neighbors of “unequal” size (e.g., Pakistan—India, Greece—Turkey, Iraq—Iran, Vietnam—China, Chad—Libya and Taiwan—China).1 Deger and Sen (1984) present optimal control and differential game models of arms races incorporating these two important

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*Author is at: Department of International Trade, Chinese Culture University, Taiwan. The author is grateful to the referees for comments on the earlier version of the paper.

1See Franck and Melese (2003).
characters in the Third World countries. They conclude that under reasonable conditions, both types of model satisfy the stability conditions. However, Desai (1995) assumes that both countries’ utility functions are linear in the level of weapon stocks. His analysis of the non-cooperative game suggests an unstable equilibrium, a result opposite of Deger and Sen.

Most studies on arms accumulation use two alternative specifications in the utility function and find the different dynamic results. Brito (1972), Simaan and Cruz (1975), and Deger and Sen (1983, 1984) assume that the utility function is non-separable between the consumption and the weapon stocks, while van der Ploeg and Zeeuw (1989, 1990) assume that the utility function is separable between the consumption and the weapon stocks. In addition, Zou (1995) and Chang et al. (1996, 2002) use two alternative specifications in the utility function and find that, in the short run, the economy could have different dynamic responses.2

Kamien and Schwartz (1991, p.272) point out that, “In particular, in the case of an optimal control problem it is a single individual’s choice of the control variable that advances the state of the system…Situations in which the joint actions of several individuals, each acting independently, either effect a common state variable or each other’s payoffs through time, are modeled as differential games.” Therefore, in the optimal control model, we believe that one country will have to predict the strategy of the other before starting its own optimization process. On the other hand, in the differential game model, the interactions between one country and its adversary become important very much. Hence, we have to derive the reaction functions of the two countries and continue to analyze its stability. However, both types of model are useful in modeling defense in the Third World countries, see Deger and Sen (1984, p.156).

We believe that the different assumptions about the utility function can lead to different dynamic adjustments. Thus, this paper assumes that the utility function is separable between the consumption and the weapon stocks. It is found that, when

2Zou (1995) and Chang et al. (1996, 2002) prove that the form of the utility function will influence the short-run dynamic responses in a dynamic optimization model. However, in the long run, the dynamic responses will not be influenced by the two alternative specifications in the utility function. For example, Zou (1995, p.371) indicates that, “…when the utility function is separable between consumption and the weapon stocks, an unanticipated rise in current military threat reduces current investment and an anticipated rise in future military threat stimulates current investment. But when the utility function is nonseparable between consumption and the weapon stocks, a current military threat may not decrease the short-run investment.”
the utility function is separable between the consumption and the weapon stocks, the military expenditures and the weapon stocks will exhibit stability for the optimal control models. Moreover, we also prove that the simplified differential game model has an unstable equilibrium.

We organize the paper as follows. In section 2 the optimal control model for the small country is set up. Section 3 does the same for the large country. Section 4 sets up the differential game model. Some concluding remarks are presented in section 5.

2 Optimal control model for the small country

As in Deger and Sen (1984), there are two countries in this model: the small country and the large country, and they are in a state of arms races (e.g., Pakistan—India, Greece—Turkey, Iraq—Iran, Vietnam—China, Chad—Libya and Taiwan—China). However, in the Third World countries, the optimal resource allocation problem between civilian and military expenditures is an important issue. Military expenditures can hinder civilian development through the channel of the budgetary crowding-out effect. Based upon this stylized fact, Deger and Sen (1984) use military expenditures as a ratio of GNP as the decision variable, with one minus this ratio giving the ratio spent on civilian development (consumption). Thus, an increase in military expenditures decreases the nation’s utility. Because the weapon stocks can provide a measure of security, an increase in the weapon stocks increases the country’s utility. The external threat (foreign weapon stocks) from its large neighbor is treated as an exogenous variable. A rise in the external threat will decrease the country’s utility. Therefore, the small country derives utility from its military expenditures as a ratio of its GNP ($m_s$), an exogenously defined level of external threat ($\theta$), and its weapon stocks as a ratio of its GNP ($s_s$). Following Deger and Sen (1983, 1984), van der Ploeg and de Zeeuw (1990), Zou (1995), and Chang et al. (1996, 2002), the small country’s instantaneous utility function $u(m_s, \theta, s_s)$ is specified as follows:

$$u_1 < 0, \ u_2 < 0, \ u_3 > 0, \ u_{11} < 0, \ u_{12} < 0, \ u_{13} = u_{23} = 0, \ u_{22} > 0. \quad \text{3}$$

The assumptions are the same as Zou (1995, p.373) equations (1) and (2), Chang et al. (1996, p.508) equation (1) and Chang et al. (2002, p.1038) equation (4).
Let $u_i$ represent the derivative of the utility function with respect to the $i^{th}$ argument. Similarly, $u_{ij}$ represents the derivative of $u_i$ w.r.t. $j^{th}$ argument. The aforementioned assumptions imply that the marginal utility from military expenditures (civilian consumption) is negative (positive) and diminishing. In addition, the marginal utility from weapon stocks is positive and diminishing, but the marginal utility from external threat is negative. The assumption $u_{iz} = u_{zi} < 0$ implies that more weapon stocks reduce (raise) the marginal utility of military expenditures (civilian consumption), whereas $u_{iz} = u_{zi} > 0$ states that a rise in the external threat raises (reduces) the marginal utility of military expenditures (civilian consumption), similarly, $u_{iz} = u_{zi} > 0$ asserts that an increase in the external threat will increase the marginal utility of small country’s weapon stocks. Moreover, the assumption $u_{iz} = u_{zi} = u_{zi} = 0$ implies that the utility function is separable between the consumption and the weapon stocks. The recent development of armament races literatures emphasizes that two alternative specifications in the utility function can lead to different dynamic adjustments. Therefore, we assume that the utility from consumption is independent of the weapon stocks; i.e., $u_{iz} = u_{zi} = u_{zi} = 0$, see Zou (1995).4

The objective of the small country is to maximize the discounted sum of instantaneous utilities over an infinite horizon:

$$W^* = \int_0^\infty e^{-\rho_1 t} u(m_i, \theta_i, s_i) dt,$$

where $\rho_1$ is the discount rate of the small country, $0 < \rho_1 < 1$. The military expenditures are entirely on arms accumulation and weapon replacement, that is,

$$\dot{s}_i = m_i - \alpha s_i,$$

where $\alpha$ is the sum of the growth rate in national income and the rate of depreciation in the weapon stocks.5 Dots represent time derivatives. Thus, the small country’s inter-temporal optimization can be summarized as:

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4We could specify the separable utility function as follows: $u(m_i, \theta_i, s_i) = \phi_0 - (1/2)\phi_2 (m_i - \phi_1/\phi_2)^2 - (1/2)\phi_4 (s_i - \theta_i - \phi_3/\phi_4)^2$, with parameters satisfying: $\phi_1 < 0$, $\phi_2 > 0$, and $\phi_4 > 0$. Please see van der Ploeg and Zeeuw (1990, p.134) equation (2). In the meanwhile, we also could use the same specification in the instantaneous utility function $v(n_t, \theta_t, s_t)$ in section 3.

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\[
\max_{\alpha} \int_0^\infty e^{-\alpha t}u(m(t), \theta(t), s(t))dt, \\
\text{s.t. } \dot{s}_i = m_i - \alpha_s s_i.
\]

In optimal control problems, variables can be divided into control variable \(m_i\) and state variable \(s_i\). The current value Hamiltonian can be written as:

\[
H^1 = u(m_i, \theta_i, s_i) + \lambda_i (m_i - \alpha s_i),
\]

where \(\lambda_i\) is the costate variable which can be interpreted as the current-value shadow price of the weapon stocks.\(^6\) The optimum conditions necessary for the optimization are:\(^7\)

\[
u_i + \lambda_i = 0, \tag{2}
\]

\[
\dot{\lambda}_i = (\rho_i + \alpha_i)\lambda_i - u_i, \tag{3}
\]

\[
\dot{s}_i = m_i - \alpha s_i. \tag{1}
\]

Equation (2) states that the marginal utility of military expenditures and the current-value shadow price of the weapon stocks are equal to zero or says that the marginal utility of civilian consumption is equal to the current-value shadow price of the weapon stocks. The familiar Euler condition is given by equation (3), which governs the optimal choice between military expenditures (civilian consumption) and weapon stocks accumulation. At the steady-state equilibrium, the economy is characterized by \(\dot{\lambda}_i = \dot{s}_i = 0\). Meanwhile, the stationary levels are represented by \(\bar{m}_i, \bar{s}_i, \bar{\lambda}_i\). Total differentiating equations (1)-(3) at the steady-state equilibrium (i.e., \(\dot{\lambda}_i = \dot{s}_i = 0\)), we can derive the following steady-state relationship:

\[
\frac{\partial \bar{m}_i}{\partial \theta_i} = \alpha_i \frac{\partial \bar{s}_i}{\partial \theta_i} > 0, \tag{4}
\]

\[
\frac{\partial \bar{s}_i}{\partial \theta_i} = -\frac{1}{\Delta} [u_{i_2} + u_{i_2} (\rho_i + \alpha_i)] > 0, \tag{5}
\]

\[
\frac{\partial \bar{\lambda}_i}{\partial \theta_i} = \frac{1}{\Delta} [u_{i_1} (u_{i_1} + \alpha_i u_{i_1}) - u_{i_2} (u_{i_1} + \alpha_i u_{i_1})] < 0. \tag{6}
\]

\(^6\)The variable \(\lambda_i\) can be interpreted as the current-value shadow price of the state variable. See Barro and Sala-i-Martin (1995, p.509-p.510).

\(^7\)Kamien and Schwartz (1991, p.174-p.184) indicate that the derivative process of the optimal control problem.
where $\Delta = (\rho_i + \alpha_i)(u_{i3} + \alpha_i u_{i2}) + u_{i2} + \alpha_i u_{i1} < 0$. Equations (4)-(6) tell us that a rise in the external threat leads to more military expenditures (less consumption), more arms accumulation, and uncertain movements in the current-value shadow price of small country’s weapon stocks. However, if the utility function is separable between the consumption and the weapon stocks (i.e., $u_{i3} = u_{i1} = u_{i2} = 0$), equation (6) definitely indicates that a rise in the variable $\theta_i$ will increase the current-value shadow price.

Using the necessary condition (i.e., equations (1)-(3)), we can derive the following differential equation:

$$
\dot{m}_i = [u_i - u_{i3}(m_i - \alpha_i s_i) + (\rho_i + \alpha_i)u_i]/u_{i3}.
$$

Equations (7) and (1) describe the dynamic behavior of the economy. In characterizing the steady-state equilibrium of the model, we get that there exists a steady-state equilibrium (i.e., $\overline{m}_i$ and $\overline{s}_i$), which satisfies $\dot{m}_i = \dot{s}_i = 0$ and is a saddle point. On the other hand, if the utility function is separable between the consumption and the weapon stocks, the steady-state equilibrium also is a saddle point. This result is identical with Deger and Sen (1984).

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1Differentiating equation (2) with respect to time and being substituted into equation (3).

2Appendix A states the stability analysis of the dynamic system.
The phase diagram for the dynamic system is given in figure 1. When the utility function is separable between the consumption and the weapon stocks, the dynamic system is also described by figure 1.\textsuperscript{10} Point $E_i$ is a saddle point equilibrium for initial level of external threat ($\theta_i$) with $CE_iD$ being the saddle path. An increased level of external threat ($\theta_i'$) will shift the new equilibrium to $E_2$ and the new saddle path is $FE_iG$.\textsuperscript{11} Hence, if the external threat increase from $\theta_i$ to $\theta_i'$, the control variable ‘jumps’ to attain $\bar{m}_i$, and then once again follows the stable trajectory to the new equilibrium $E_i$. This result is identical with Deger and Sen (1984) model where “overshooting” takes place, i.e., the military burden is higher than its equilibrium value at greater external threat but then steadily falls. Many Third World countries spend 3 to 4 per cent of their GDP on the military. Namely, they allocate 20 to 25 per cent of central government spending to defense sector. The military spending often exceeds health and education expenditures, see Deger and Sen (1995). Our result can explain the aforementioned phenomenon that the military burdens of the Third World countries always are large. However, Desai (1995) assumes that the utility function is linear in weapon stocks and proves that overshooting doesn’t take place. Namely, as the marginal utility of weapon stocks is positive and remains unchanged, Desai’s outcome can not explain the aforementioned phenomenon.

3 Optimal control model for the large country

The large country faces a similar allocation problem between civilian and military expenditures. As in Deger and Sen (1984), we assume that the large country is more powerful and wishes to maintain some form of regional superiority over the regional block. Thus, it will feel secure if large country’s weapon stock which is relative to total military expenditures in the region is high. Namely, the large country measures its security by calculating its weapon stocks as a ratio of the total military expenditures in the region. An increase in this variable increases the nation’s utility. Once again, military expenditures are viewed as a necessary evil and the large country’s utility decreases as the military expenditures as a ratio of the total military

\textsuperscript{10}Appendix B states the slopes of all loci.
\textsuperscript{11}Appendix B states the impact of the external threat on all loci.
expenditures in the region rises. The external threat from its small neighbor is treated as an exogenous variable. A rise in the external threat will decrease the country’s utility. Therefore, the large country derives utility from its military expenditures as a ratio of the total military expenditures in the region ($n_t$), an exogenously defined level of external threat ($\theta_t$), and its weapon stocks as a ratio of the total military expenditures in the region ($t_t$). Once again, the large country’s instantaneous utility function $v(n_t, \theta_t, t_t)$ is specified as follows:

$$v_i < 0, \quad v_2 < 0, \quad v_3 > 0, \quad v_{11} < 0, \quad v_{33} < 0,$$

$$v_{13} = v_{33} \leq 0, \quad v_{12} = v_{23} \geq 0, \quad v_{32} = v_{22} > 0.$$

Let $v_i$ represent the derivative of the utility function with respect to the $i^{\text{th}}$ argument. Similarly, $v_j$ represents the derivative of $v_i$ w.r.t. $j^{\text{th}}$ argument. The aforementioned assumptions are similar to the assumptions of the small country’s instantaneous utility function. Hence, $v_{13} = v_{31} = v_{12} = v_{23} = 0$ implies that the utility function is separable between the consumption and the weapon stocks.

The objective of the large country is to maximize the discounted sum of instantaneous utilities over an infinite horizon:

$$W^* = \int_0^\infty e^{-\rho t} v(n_t, \theta_t, t_t) dt ,$$

where $\rho$ is the discount rate of the large country, $0 < \rho < 1$. The military expenditures are entirely on arms accumulation and weapon replacement, that is,

$$i_t = n_t - \alpha_x t_t,$$

where $\alpha_x$ is the sum of the growth rate in total military expenditures in the region and the rate of depreciation in the weapon stocks. Thus, the large country’s inter-temporal optimization can be summarized as:

$$\max_{n_t,\theta_t, t_t} \int_0^\infty e^{-\rho t} v(n_t, \theta_t, t_t) dt ;$$

s.t. $i_t = n_t - \alpha_x t_t$.

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The variable $n_2$ is viewed as control variable and $t_2$ as state variable. Therefore, the current value Hamiltonian can be written as:

$$H^2 = v(n_2, \theta_2, t_2) + \lambda_z (n_2 - \alpha_2 t_2),$$

where $\lambda_z$ is the costate variable which can be interpreted as the current-value shadow price of the weapon stocks. The optimum conditions necessary for the optimization are:

- $v_i + \dot{\lambda}_z = 0$,
- $\dot{\lambda}_z = (\rho_z + \alpha_z) \lambda_z - v_z$,
- $\dot{t}_z = n_z - \alpha_z t_z$.

Once again, at the steady-state equilibrium, the economy is characterized by $\dot{\lambda}_z = \dot{t}_z = 0$. Using a method similar to the one used for the small country, it is easy to show that the steady state is indeed a saddle point. Similarly, if the utility function is separable between the consumption and the weapon stocks, the steady state also is a saddle point. Figure 2 gives the phase portrait for this equilibrium. Point $E$ is a saddle point and line $QER$ is a saddle path. The dynamic adjusting process is similar to the case of the small country in the section 2. Hence, in our large country
model, “overshooting” also takes place.

4 A differential game model

In this section, the variables of the external threat will be treated endogenously in
the differential game model. Following Deger and Sen (1984), we use $n_1$ as a
measure of the external threat faced by the small country, and $m_1$ as a measure of
the external threat faced by the large country. Deger and Sen (1984, p.163) point out
that, “Since these aggressive neighbors do not like to cooperate with each other, a
Pareto solution is immediately ruled out…However, for most LDC’s, the conditions
required for the Stackelberg strategy may be inapplicable. Due to political instability
and changes in regimes, it is difficult for the large country to declare and maintain its
strategies. Similarly, the small country will find it difficult to believe the
aforementioned declaration, either because its own government has changed
meanwhile, or there has been political upheavals in the large country.” As mentioned
earlier, we have a non-cooperative differential game of the closed loop Nash
equilibrium concept. Thus, the current value Hamiltonians for the two countries
can be written as:

$$H^1 = u(m_1, n_1, s_1) + \lambda_1(m_1 - \alpha_1 s_1) + \lambda_2(n_1 - \alpha_2 f_2),$$
$$H^2 = v(n_2, m_2, t_2) + \lambda_3(m_2 - \alpha_3 s_2) + \lambda_4(n_2 - \alpha_4 f_2),$$

where $\lambda_1, \lambda_2, \lambda_3,$ and $\lambda_4$ are the costate variables. The variables $m_1$ and $n_1$
are viewed as control variables. Meanwhile, the variables $s_1$ and $t_2$ are also
viewed as state variables. Hence, the relationship of $u_{i_1} = u_{i_2} = v_{i_1} = v_{i_2} = 0$
states that two countries’ utility functions are separable between the consumption and the
weapon stocks in differential game model. The necessary conditions for the
optimization are:

$$u_i + \lambda_{i_1} = 0,$$  \hspace{1cm} (8)
$$v_i + \lambda_{i_2} = 0,$$  \hspace{1cm} (9)
$$\dot{\lambda}_{i_1} = (\rho_i + \alpha_i)\lambda_{i_1} - u_i,$$  \hspace{1cm} (10)

\[13\] The Nash equilibrium closed loop strategies mean that the control at each point in time is a
function of both time and the state of the system (see e.g., Simaan and Cruz, 1975; Basar and Olsder,
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\[ \dot{\lambda}_{22} = (\rho_2 + \alpha_2)\lambda_{22} - v_i. \]  

(11)

Because \( \lambda_{i2} \) and \( \lambda_{2i} \) do not appear in equations (8)-(11), we will exclude these two variables in the rest of this paper. Thus, differentiating equations (8)-(9) with respect to time and being substituted into equations (10) and (11), we can derive the following differential equation:

\[ \begin{align*}
\dot{m}_i &= \left[ (\rho_1 + \alpha_1)u_i + u_i \right]v_i - \left[ (\rho_2 + \alpha_2)v_i + v_i \right]u_{12}, \\
\dot{n}_i &= \left[ (\rho_2 + \alpha_2)v_i + v_i \right]u_i - \left[ (\rho_1 + \alpha_1)u_i + u_i \right]v_{12}.
\end{align*} \]

(12)

(13)

The Nash optimal control strategies of the two countries must satisfy (12) and (13). At the steady-state equilibrium, the economy is characterized by \( \dot{m}_i = \dot{n}_i = 0 \) and \( m_i, n_i \) are at their stationary level, namely \( \overline{m}_i, \overline{n}_i \). From equations (12) and (13), it is clear that \( \dot{m}_i = \dot{n}_i = 0 \) if the relationship of \( (\rho_1 + \alpha_1)u_i + u_i = (\rho_2 + \alpha_2)v_i + v_i = 0 \) holds. This result implies that the economy is always steady state if the marginal rate of substitution (MRS) of the consumption and the weapon stocks equals \( 1/(\rho_i + \alpha_i) \) \( i = 1,2 \) in the two countries.\(^{14}\)

In order to analyze the dynamic system described by equations (12) and (13), we use the Taylor series expansion near the steady-state point and keep the linear terms. Deger and Sen (1984) assume that two countries’ utility functions are not separable between their arguments and prove that this steady state point is a saddle point. However, we assume that two countries’ utility functions are separable between the consumption and the weapon stocks (i.e., \( u_{i1} = u_{i2} = v_{i1} = v_{i2} = 0 \)) and will show that saddle-point equilibrium is not possible for the above dynamic system.

Therefore, the Jacobian for the above dynamic system can be written as:

\[
J = \begin{bmatrix}
\frac{\partial \dot{m}_1}{\partial m_1} & \frac{\partial \dot{m}_1}{\partial n_1} \\
\frac{\partial \dot{m}_2}{\partial m_2} & \frac{\partial \dot{m}_2}{\partial n_2} \\
\frac{\partial \dot{n}_1}{\partial m_1} & \frac{\partial \dot{n}_1}{\partial n_1} \\
\frac{\partial \dot{n}_2}{\partial m_2} & \frac{\partial \dot{n}_2}{\partial n_2}
\end{bmatrix},
\]

\(^{14}\)The MRS of the consumption and the weapon stocks in the two countries are defined as \(-u_i/u_i\) and \(-v_i/v_i\).
where

\[
\begin{align*}
\frac{\partial \dot{m}_1}{\partial m_1} &= (\rho_1 + \alpha_1)u_{i_1}v_{i_1} - (\rho_2 + \alpha_2)v_{i_2}u_{i_2}, \\
\frac{\partial \dot{m}_1}{\partial m_2} &= u_{i_1}v_{i_1} - v_{i_1}u_{i_1}, \\
\frac{\partial \dot{m}_2}{\partial m_1} &= (\rho_1 + \alpha_1)u_{i_2}v_{i_1} - (\rho_2 + \alpha_2)v_{i_2}u_{i_1}, \\
\frac{\partial \dot{m}_2}{\partial m_2} &= u_{i_1}v_{i_1} - v_{i_1}u_{i_1}, \\
\frac{\partial \dot{m}_1}{\partial n_1} &= -(\rho_1 + \alpha_1)u_{i_2}v_{i_2} + (\rho_2 + \alpha_2)v_{i_2}u_{i_1}, \\
\frac{\partial \dot{m}_2}{\partial n_2} &= -(\rho_1 + \alpha_1)u_{i_2}v_{i_2} + (\rho_2 + \alpha_2)v_{i_2}u_{i_1}. \\
\frac{\partial \dot{n}_1}{\partial n_1} &= -(\rho_1 + \alpha_1)u_{i_2}v_{i_2} + (\rho_2 + \alpha_2)v_{i_2}u_{i_1}, \\
\frac{\partial \dot{n}_2}{\partial n_2} &= -(\rho_1 + \alpha_1)u_{i_2}v_{i_2} + (\rho_2 + \alpha_2)v_{i_2}u_{i_1}.
\end{align*}
\]

Since \( \det J = (\rho_1 + \alpha_1)(\rho_2 + \alpha_2) > 0 \), trace \( J = \rho_1 + \alpha_1 + \rho_2 + \alpha_2 > 0 \) and \( (\text{trace } J)^2 \geq 4 \det J \), the equilibrium point is an unstable node, see Chiang (1984). This result is different from Deger and Sen (1984) model which have a stable equilibrium, but identical with Desai (1995) model.

The concept of stability means the crisis stability. The unstable equilibrium may lead to the outbreak of war, see Brito and Intriligator (1995). We observe that the end of the Cold War exclude tension between the West and the East, but regional conflicts have continued to exist, including the Arab-Israel conflict, the Iran-Iraq conflict, the Iraq-Kuwait conflict (Gulf War), the Vietnam-China conflict (Vietnam War), and the India-Pakistan conflict (see e.g., Brito and Intriligator, 1995; Shieh et al., 2002). However, Subrahmanyam (1988, p.33) also points out that, “…of more than 170 major inter- and intra-state conflicts that have occurred since the end of World War II, over 160 have occurred in the developing world.” Hence, our result can be viewed as the explanation of the phenomenon of the outbreak of war.

5 Conclusion

This paper examines the stability of armaments races for the optimal control model and the differential game model in the Third World countries. We find that the military expenditures and the weapon stocks will exhibit stability for the optimal control models. Meanwhile, in our model, overshooting takes place. This outcome can explain the general phenomenon that the military burdens of the Third World countries always are large. Finally, we prove that the differential game model has an unstable equilibrium. However, the unstable paths are believed to lead to war. Our
result can be regarded as a theoretical explanation for the existence of regional conflicts, including the Arab-Israel conflict, the Iran-Iraq conflict, the Iraq-Kuwait conflict, the Vietnam-China conflict, and the India-Pakistan conflict.

Appendix A

We now explore the dynamic behavior of the economy described by equations (7) and (1). In order to examine the local stability of the dynamic system, we rewrite equations (7) and (1) as follows

\[
\dot{m}_t = G(m_t, \theta, s_t), \tag{A.1}
\]

\[
\dot{s}_t = K(m_t, \theta, s_t). \tag{A.2}
\]

Linearizing equations (A.1) and (A.2) around the steady-state equilibrium, we have

\[
\begin{bmatrix}
\dot{m}_t \\
\dot{s}_t
\end{bmatrix} =
\begin{bmatrix}
G_m & G_s \\
K_m & K_s
\end{bmatrix}
\begin{bmatrix}
m_t - \bar{m} \\
s_t - \bar{s}
\end{bmatrix}, \tag{A.3}
\]

where

\[
G_m = \rho_i + \alpha_i > 0, \\
G_s = [u_i + (\rho_i + \alpha_i)u_{i,t} + \alpha_i u_{i,t}]/u_i > 0, \\
K_m = 1, \\
K_s = -\alpha_i < 0.
\]

Let the parameter \( \Omega \) represent the determinant of this coefficient matrix. We get

\[
\Omega = -[\alpha_i(\rho_i + \alpha_i) + [u_i + (\rho_i + \alpha_i)u_{i,t} + \alpha_i u_{i,t}]/u_i] < 0. 
\]

Hence, one of the eigenvalues of the coefficient matrix is positive and the other is negative. Namely, the steady-state equilibrium is a saddle point. On the other hand, if the utility function is separable between the consumption and the weapon stocks, we have

\[
\Omega = -[\alpha_i(\rho_i + \alpha_i) + u_{i,t}/u_i] < 0
\]

and the steady-state equilibrium also is a saddle point.

Appendix B

We now analyze the slopes of loci \( \dot{m}_t = 0 \) and \( \dot{s}_t = 0 \) in figure 1. It is clear from
equation (A.3) that the slopes of loci $\dot{m}_1 = 0$ and $\dot{s}_i = 0$ are

\[
\frac{\partial \dot{m}_1}{\partial \dot{s}_i} \bigg|_{i=0} = -G_\alpha / G_\alpha < 0, \\
\frac{\partial \dot{m}_1}{\partial \dot{s}_i} \bigg|_{i=0} = -K_\alpha / K_\alpha > 0.
\]

Next, we illustrate the slope of saddle path. We assume that the parameters $\eta_1$ and $\eta_2$ are the two characteristic roots that satisfy dynamic equations (A.1) and (A.2). If $\eta_1 < 0 < \eta_2$ is assumed, the slope of the stable saddle path is

\[
\frac{\partial \dot{m}_1}{\partial \dot{s}_i} \bigg|_{\text{saddle path}} = -G_\alpha / (G_\alpha - \eta_1) < 0,
\]

Evidently, the convergent saddle path is always downward sloping and is flatter than the $\dot{m}_1 = 0$ locus.

Finally, we study the impact of the external threat on $\dot{m}_1 = 0$ and $\dot{s}_i = 0$ loci. From equations (A.1) and (A.2), we have

\[
\frac{\partial \dot{m}_1}{\partial \theta_i} \bigg|_{i=0} = -G_\alpha / G_\alpha > 0, \\
\frac{\partial \dot{m}_1}{\partial \theta_i} \bigg|_{i=0} = -K_\alpha / K_\alpha = 0,
\]

where

\[
G_\alpha = [u_{\alpha_2} + (\rho_1 + \alpha_1)u_{\alpha_2}] / u_{\alpha_2} < 0, \\
K_\alpha = 0.
\]

Evidently, a rise of external threat from $\alpha$ to $\alpha'$ will lead to shift the $\dot{m}_1 = 0(\theta_i)$ locus upward to the $\dot{m}_1 = 0(\theta_i')$ locus, while the $\dot{s}_i = 0$ locus remains intact.

References


