Irrelevance of Conjectural Variations in Duopoly under Relative Profit Maximization and Consistent Conjectures

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We study the equilibrium with the quantity setting behavior and price setting behavior of firms in a duopoly under relative profit maximization with constant conjectural variations, and arrive at the following results. 1) Conjectural variations of firms are irrelevant to the equilibrium of a duopoly. 2) Quantity setting behavior and price setting behavior are equivalent to any conjectural variation for each firm. 3) Any pair of conjectural variations of firms which satisfies some relation is consistent. In particular, if firms have the same cost functions or the cost functions are linear, and both firms determine the outputs or both firms determine the prices, any conjectural variations which are common to both firms are consistent. Therefore, there are multiple consistent conjectures.

**Keywords:** duopoly, relative profit maximization, conjectural variation, consistent conjecture

**JEL classification:** D43, L13, L21

1 Introduction

We study the equilibrium with the quantity setting behavior and price setting
behavior of firms in a duopoly under relative profit maximization with constant conjectural variations. Conjectural variations in oligopoly and their consistency have been studied in many papers such as Bresnahan (1981), Bresnahan (1983), Robson (1983), Perry (1982), Boyer and Moreaux (1983), Tanaka (1985) and Tanaka (1988). Bresnahan (1981) defines a consistent conjectural equilibrium (CCE) and has shown the existence of CCE in the case where the demand and cost functions are linear and the conjectural variations are polynomial. Bresnahan (1983) has shown the existence of CCE when demand functions are quadratic using a formulation by Robson (1983).

In Perry (1982) it was shown that in a duopoly producing a homogeneous good under constant marginal costs the competitive conjecture is the only consistent conjecture. In Tanaka (1985) it was shown that in a free entry oligopoly only the competitive conjecture is consistent. Klemperer and Meyer (1988) presented an interpretation of Bresnahan’s formulation of consistent conjectures as a dominant strategy equilibrium and showed the multiplicity of consistent conjectures. Kamien and Schwartz (1983), among others, showed that if the demand function is linear, constant consistent conjectural variations in quantity correspond to constant consistent conjectural variations in price in the sense that the same equilibrium price and quantity will be attained if either conjectural variation is constant.

We consider a duopoly in which firms produce differentiated substitutable goods under linear demand functions and general cost functions so as to maximize their relative profits, and we consider constant conjectural variations in regard to outputs and prices. We consider three frameworks:
1. (Quantity setting framework) Both firms determine their outputs (quantities).
2. (Price setting framework) Both firms determine the prices of their goods.
3. (Mixed framework) One firm determines its output, and the other firm determines the price of its good.

We show the following results:
1. The conjectural variations of firms are irrelevant to the equilibrium of a duopoly.
2. The equilibrium outputs and prices in all three frameworks are equal, that is, the quantity setting behavior and price setting behavior are equivalent to any conjectural variation for each firm.
3. Any pair of conjectural variations of firms which satisfies some relation is
consistent and, in particular, if firms have the same cost functions or the cost functions are linear, and both firms determine their outputs or both firms determine their prices, any conjectural variations which are common to both firms are consistent. Therefore, there are multiple consistent conjectures.


In Vega-Redondo (1997), it is argued that, in a homogeneous good case, if firms maximize relative profits, a competitive equilibrium can be induced. However, in the case of differentiated goods, the result under relative profit maximization is different from the competitive result. Miller and Pazgal (2001) have shown the equivalence of a price strategy and quantity strategy in a delegation game when owners of firms that control managers of firms seek to maximize an appropriate combination of absolute and relative profits. However, in their analyses the owners of the firms themselves still seek to maximize the absolute profits of their firms. On the other hand, we do not consider the delegation problem, and assume that the owners of firms seek to maximize their relative profits.

We consider that seeking a relative profit or utility is based on human nature. Even if a person earns much money, if his brother/sister or close friend earns more money than him, he is not sufficiently happy and may be disappointed. On the other hand, even if he is very poor, if his neighbor is poorer, he may be consoled by that fact. As for the behavior of firms, we believe that firms in an industry not only seek to improve their own performance but also seek to outperform their rival firms. TV audience ratings and the competition for market share among breweries, automobile manufacturers, convenience store chains and mobile-phone carriers, especially in Japan, are examples of such behavior by firms.

In Section 3 we consider a case where both firms determine their outputs (quantities). In Section 4 we consider a case where both firms determine the prices of their goods. Finally, in Section 5 we consider a case where one firm determines its output and the other firm determines the price of its good.

2 The Model
There are two firms, A and B, that produce differentiated substitutable goods. Denote the output and price of the good of Firm A by \( x_A \) and \( p_A \), and the output and price of the good of Firm B by \( x_B \) and \( p_B \). The cost functions of Firm A and B are, respectively, denoted by \( c_A(x_A) \) and \( c_B(x_B) \), and the marginal cost functions of Firm A and B are denoted by \( c'_A(x_A) \) and \( c'_B(x_B) \). They may be different. The constant conjectural variations of Firm A and B in the quantity setting framework are denoted, respectively, by \( \delta_A \) and \( \delta_B \). In the price setting framework they are denoted, respectively, by \( \eta_A \) and \( \eta_B \). Assume that \(-1 < \delta_A < 1\), \(-1 < \delta_B < 1\), \(-1 < \eta_A < 1\) and \(-1 < \eta_B < 1\). In the mixed framework the conjectural variations are denoted by \( \zeta_A \) and \( \zeta_B \). We do not assume \(-1 < \zeta_A < 1\) nor \(-1 < \zeta_B < 1\).

The inverse demand functions of the goods produced by firms are:

\[
p_A = a - x_A - bx_A, \tag{1}
\]

and

\[
p_B = a - x_B - bx_B, \tag{2}
\]

where \(0 < b < 1\). \(x_A\) represents the demand for the good of Firm A, and \(x_B\) represents the demand for the good of Firm B. The prices of the goods are determined so that the demand of consumers for each firm’s good and the supply of each firm are equilibrated.

The ordinary demand functions for the goods of firms are obtained from these inverse demand functions as follows,

\[
x_A = \frac{1}{1-b^2}[(1-b)a - p_A + bp_A],
\]

and

\[
x_B = \frac{1}{1-b^2}[(1-b)a - p_B + bp_B].
\]

Next consider a case where Firm A determines the price of its good and Firm B determines its output. Then, from (1) the ordinary demand function for Firm A is:

\[
x_A = a - p_A - bx_A, \tag{3}
\]
Substituting this into (2), the inverse demand function for Firm B is:

\[ p_B = (1 - b)a - (1 - b^2)x_a + bp_A. \]  

(4)

3 Quantity Setting Framework

Assume that the strategic variable of each firm is the output of its good. The relative profit of Firm A (or B) is the difference between its profit and the profit of Firm B (or A). We denote the relative profit of Firm A by \( \Pi_A \) and that of Firm B by \( \Pi_B \).

They are written as follows,

\[ \Pi_A = \pi_A - \pi_B = (a - x_A - bx_A)x_A - (a - x_B - bx_B)x_B - c_A(x_A) + c_B(x_B) \]

\[ = a(x_A - x_A^2) - x_A^2 + x_A^3 - c_A(x_A) + c_B(x_B), \]

and

\[ \Pi_B = \pi_B - \pi_A = (a - x_B - bx_B)x_B - (a - x_A - bx_A)x_A - c_B(x_B) + c_A(x_A) \]

\[ = a(x_B - x_B^2) - x_B^2 + x_B^3 - c_B(x_B) + c_A(x_A). \]

Firm A determines its output so as to maximize its relative profit assuming that the reaction of the output of Firm B to the output of Firm A is:

\[ \frac{\partial x_A}{\partial x_B} = \delta_A; \]

and Firm B determines its output so as to maximize its relative profit assuming that the reaction of the output of Firm A to the output of Firm B is:

\[ \frac{\partial x_B}{\partial x_A} = \delta_B. \]

The conditions for the relative profit maximization of Firms A and B are, respectively,

\[ a - c_A'(x_A) - 2x_A - (a - c_A'(x_A))\delta_A + 2\delta_A x_B = 0, \]  

(5)

and

\[ a - c_B'(x_B) - 2x_B - (a - c_B'(x_B))\delta_B + 2\delta_B x_A = 0. \]  

(6)
From (5) and (6) we have:

\[ (1-\delta_A\delta^*_A)(a-c'_A(x_A)) = 2(1-\delta_A\delta^*_A)x_A. \]

and

\[ (1-\delta_B\delta^*_B)(a-c'_B(x_B)) = 2(1-\delta_B\delta^*_B)x_B. \]

Then, the equilibrium outputs are:

\[ x_A = \frac{a-c'_A(x_A)}{2}. \]

and

\[ x_B = \frac{a-c'_B(x_B)}{2}. \]

They do not depend on the values of \( \delta_A \) or \( \delta_B \). Thus, the conjectural variations of firms are irrelevant to the equilibrium outputs under relative profit maximization in the quantity setting framework.

The equilibrium prices are obtained as follows:

\[ p_A = \frac{(1-b)a + c'_A(x_A) + b c'_A(x_B)}{2}. \]

and

\[ p_B = \frac{(1-b)a + c'_B(x_B) + b c'_B(x_A)}{2}. \]

Again from (5) and (6) the real reaction of the output of Firm A to the output of Firm B and the real reaction of the output of Firm B to the output of Firm A are obtained as follows,

\[ \frac{\partial x_A}{\partial x_B} = \frac{2 + c''_A(x_A)}{2 + c''_B(x_B)} \delta^*_A. \]

and

\[ \frac{\partial x_B}{\partial x_A} = \frac{2 + c''_B(x_B)}{2 + c''_A(x_A)} \delta^*_B. \]
The conditions for the consistency of conjectural variations are:

\[ \frac{2 + c_s' (x_a)}{2 + c_s' (x_a)} \delta_a = \delta_a, \]

and

\[ \frac{2 + c_s' (x_b)}{2 + c_s' (x_b)} \delta_b = \delta_b. \]

They are the same equations. Therefore, any pair of conjectural variations, \( \delta_a \) and \( \delta_b \), which satisfies the condition:

\[ \frac{\delta_a}{\delta_b} = \frac{2 + c_s' (x_a)}{2 + c_s' (x_b)}, \quad (7) \]

is consistent. If the firms have the same cost functions, the equilibrium outputs of the firms are equal, and (7) is reduced to:

\[ \delta_a = \delta_b. \quad (8) \]

In addition, if the cost functions are linear, the second-order derivatives of the cost functions are zero, and we obtain (8). Thus, if the firms have the same cost functions, or the cost functions are linear, any common conjectural variations are consistent.

4 Price Setting Framework

In this section we assume that the strategic variable of each firm is the price of its good. Similar to the previous section, the relative profits of Firm A and B are denoted by \( \Pi_A \) and \( \Pi_B \). They are written as follows,

\[ \Pi_A = \pi_A - \pi_B = \frac{1}{1-b^2} [(1-b)a(p_A-p_B)-p_A^2+p_B^2]c_A(x_a)+c_B(x_b), \]

and
Firm A determines the price of its good so as to maximize its relative profit assuming that the reaction of the price of the good of Firm B to the price of the good of Firm A is:

$\frac{\partial p_A}{\partial p_B} = \eta_A$.

and Firm B determines the price of its good so as to maximize its relative profit assuming that the reaction of the price of the good of Firm A to the price of the good of Firm B is:

$\frac{\partial p_B}{\partial p_A} = \eta_B$.

The conditions for profit maximization of Firms A and B are, respectively,

\[ (1-b)a - 2p_A + c_s(x_A) + bc'_s(x_A) \]
\[ -[(1-b)a - 2p_B + c_s(x_B) + bc'_s(x_B)] \eta_A = 0, \]  
(9)

and

\[ (1-b)a - 2p_B + c_s(x_B) + bc'_s(x_B) \]
\[ -[(1-b)a - 2p_A + c_s(x_A) + bc'_s(x_A)] \eta_B = 0. \]  
(10)

From (9) and (10) we have:

\[ (1-\eta_A\eta_B)(1-b)a + c_s(x_A) + bc'_s(x_A) - 2(1-\eta_A\eta_B)p_A = 0. \]
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and

\[(1-\eta_i\eta_j)[(1-b)a + c'(x_a) + bc'(x_j)] - 2(1-\eta_i\eta_j)p_j = 0.\]

Then, the equilibrium prices are:

\[p_a = \frac{(1-b)a + c'(x_a) + bc'(x_j)}{2} ,\]

and

\[p_b = \frac{(1-b)a + c'(x_a) + bc'(x_j)}{2} .\]

They do not depend on the values of \(\eta_i\) or \(\eta_j\). Thus, the conjectural variations of firms are irrelevant to the equilibrium prices under relative profit maximization in the price setting framework. The equilibrium outputs are obtained as follows:

\[x_a = \frac{a - c'(x_a)}{2} .\]

and

\[x_b = \frac{a - c'(x_b)}{2} .\]

Therefore, the equilibrium prices and outputs under relative profit maximization in the price setting framework are equal to those in the quantity setting framework.

Again from (9) and (10) the real reaction of the price of the good of Firm A to the price of the good of Firm B and the real reaction of the price of the good of Firm B to the price of the good of Firm A are obtained as follows,

\[\frac{\partial p_\Delta}{\partial p_b} = \frac{2(1-b^2) + c'(s(x_a) - b^2c'(s(x_b))))}{2(1-b^2) + c'(s(x_a) - b^2c'(s(x_b))) - b\eta_s(c^*(x_a) - c^*(s(x_b)))} .\]

and

\[\frac{\partial p_a}{\partial p_b} = \frac{2(1-b^2) + c'(s(x_a) - b^2c'(s(x_b))))}{2(1-b^2) + c'(s(x_a) - b^2c'(s(x_b))) - b\eta_s(c^*(s(x_b) - c^*(s(x_b)))} .\]
In these calculations we use the ordinary demand functions. The conditions for the consistency of the conjectural variations are:

\[
\frac{[2(1-b^2)^{a} + c^*\sigma(x) - b^*c^*\sigma(x)]\eta_a - b(c^*\sigma(x) - c^*\sigma(x))}{2(1-b^2)^{a} + c^*\sigma(x) - b^*c^*\sigma(x) - b\eta_a(c^*\sigma(x) - c^*\sigma(x))} = \eta_b,
\]

and

\[
\frac{[2(1-b^2)^{a} + c^*\sigma(x) - b^*c^*\sigma(x)]\eta_b - b(c^*\sigma(x) - c^*\sigma(x))}{2(1-b^2)^{a} + c^*\sigma(x) - b^*c^*\sigma(x) - b\eta_b(c^*\sigma(x) - c^*\sigma(x))} = \eta_a.
\]

They are the same equations, and are reduced to:

\[
[2(1-b^2)^{a} + c^*\sigma(x) - b^*c^*\sigma(x)]\eta_a - b(c^*\sigma(x) - c^*\sigma(x)) + b\eta_a c^*\sigma(x) = [2(1-b^2)^{a} + c^*\sigma(x) - b^*c^*\sigma(x)]\eta_b + b(c^*\sigma(x) - c^*\sigma(x)) + b\eta_b c^*\sigma(x).
\]

Therefore, any pair of conjectural variations, \( \eta_a \) and \( \eta_b \), which satisfies (11) is consistent. If the firms have the same cost functions, then the equilibrium outputs of Firms A and B and the equilibrium prices of the goods of Firms A and B are, respectively, equal, and (11) is reduced to:

\[
\eta_a = \eta_b.
\]

In addition, if the cost functions are linear, the second-order derivatives of the cost functions are zero, and we obtain (12). Thus, if the firms have the same cost functions, or the cost functions are linear, any common conjectural variations are consistent.

5 Mixed Framework

In this section we assume that the strategic variable of Firm A is the price of its good and the strategic variable of Firm B is its output. Similar to the previous sections, the relative profits of Firms A and B are denoted by \( \Pi_a \) and \( \Pi_b \). Using (3) and (4), they are written as follows,

\[
\Pi_a = \pi_a - \pi_b = (a - p_a - bx_a)p_a - [(1-b)a - (1-b)c]x_a - c(x) = \pi_a + c(x),
\]

\[
\Pi_b = \pi_b - \pi_a = b\pi_a + \pi_a + (b-bc)\Pi_a - c(x) = \pi_b + c(x).
\]
and
\[ \Pi_a = \pi_a - \pi_A = [(1-b)a - (1-b')x_a + bp_a]x_a \\
- (a - p_a - bx_a)p_a - c_a(x_a) + c_a(x_a), \]

with
\[ x_a = a - p_a - bx_a. \]

Firm A determines the price of its good so as to maximize its relative profit assuming that the reaction of the output of the good of Firm B to the price of the good of Firm A is:
\[ \frac{\partial x_a}{\partial p_a} = \zeta_a, \]

and Firm B determines its output so as to maximize its relative profit assuming that the reaction of the price of the good of Firm A to the output of Firm B is:
\[ \frac{\partial p_a}{\partial x_a} = \zeta_a. \]

The conditions for the relative profit maximization of Firms A and B are, respectively,
\[ a - 2p_a - 2bx_a + c'_a(x_a) + [-2bp_a - (1 - b)a + 2(1 - b')x_a + c'_a(x_a)] \zeta_a = 0. \] (13)

and
\[ -2(1 - b')x_a + (1 - b)a + 2bp_a - c'_a(x_a) - bc_a(x_a) \]
\[ + [-a + 2bx_a + 2p_a - c_a(x_a)] \zeta_a = 0. \] (14)

From (13) and (14) we have:
\[ (1 + b\zeta_a)[2p_a - c'_a(x_a)] = 2[(1 - b')\zeta_a - b]x_a + c'_a(x_a)\zeta_a \]
\[ + a - (1 - b)a\zeta_a, \]

and
\[ (b + c_a)[2p_a - c'_a(x_a)] = 2[(1 - b') - b\zeta_a]x_a + a\zeta_a - (1 - b)a + c'_a(x_a). \]
Then, the equilibrium price of the good of Firm A is:

\[ p_A = \frac{(1-b)a + c'_A(x_A) + b c'_A(x_B)}{2}. \]

and the equilibrium output of Firm B is:

\[ x_B = \frac{a - c'_B(x_B)}{2}. \]

They do not depend on the values of \( \zeta_A \) or \( \zeta_B \). Thus, the conjectural variations of firms are irrelevant to the equilibrium prices and outputs under relative profit maximization in the mixed framework. The equilibrium output of Firm A and the equilibrium price of the good of Firm B are obtained as follows.

\[ x_A = \frac{a - c'_A(x_A)}{2}. \]

and

\[ p_B = \frac{(1-b)a + c'_A(x_A) + b c'_A(x_B)}{2}. \]

Therefore, we find that the equilibrium prices and outputs under relative profit maximization in the mixed framework are equal to those in the quantity setting framework and the price setting framework.

Again from (13) and (14) the real reaction of the price of the good of Firm A to the output of the good of Firm B and the real reaction of the output of Firm B to the price of the good of Firm A are obtained as follows,

\[ \frac{\partial p_A}{\partial x_B} = \frac{2[(1-b)\zeta_A - b] + \zeta_A c^*(x_B) - b(1+b\zeta_A)c^*(x_A)}{(1+b\zeta_A)(2+c^*(x_A))}. \]

and

\[ \frac{\partial x_A}{\partial p_B} = \frac{(b + \zeta_A)(2+c^*(x_A))}{2[(1-b) - b\zeta_A] + c^*(x_B) - b(b + \zeta_A)c^*(x_A)}. \]

Therefore, the conditions for the consistency of conjectural variations are:
We find that they are the same conditions and are reduced to:

\[
\frac{2[(1-b^2)\zeta_s - b] + \zeta_s c^{*}_a(x_a) - b(1+b\zeta_s)c^{*}_a(x_a)}{(1+b\zeta_s)(2+c^{*}_a(x_a))} = \zeta_a.
\]

and

\[
\frac{(b + \zeta_s)(2 + c^{*}_a(x_a))}{2[(1-b^2) - b\zeta_s] + c^{*}_a(x_a) - b(b + \zeta_s)c^{*}_a(x_a)} = \zeta_a.
\]

We find that they are the same conditions and are reduced to:

\[
2[(1-b^2)\zeta_s - b] + \zeta_s c^{*}_a(x_a) - b(1+b\zeta_s)c^{*}_a(x_a) - (1+b\zeta_s)(2+c^{*}_a(x_a))\zeta_a = 0.
\] (15)

Thus, any pair of conjectural variations which satisfies (15) is consistent under relative profit maximization in the mixed framework.

6 Concluding Remarks

In this paper, assuming constant conjectural variations, we have shown the irrelevance of conjectural variations to the equilibrium outputs and prices in duopoly under relative profit maximization, and that there are multiple consistent conjectures. We hope to generalize these results to the case of non-constant conjectural variations.

In Tanaka (2013) it was shown that under relative profit maximization Cournot and Bertrand equilibria in duopoly are equivalent. The result of this paper is an extension of that result to a duopoly with arbitrary conjectural variations. The equivalence of Cournot and Bertrand equilibria under relative profit maximization, however, may not hold in oligopoly with more than two firms.

A game of relative profit maximization in a duopoly is a two-person zero-sum game in which the payoff of one player is the opposite of the payoff of the other player. It seems to be a reason why conjectural variations are irrelevant to the equilibrium of the duopoly under relative profit maximization. A game of relative profit maximization in an oligopoly is a multi-person zero-sum game. If the number of players is larger than two, the payoff of one player is not the opposite of the payoff of another player. Thus, probably in an oligopoly the irrelevance of conjectural variations does not hold unless the oligopoly is symmetric (all firms
have the same cost functions).

References


